Math 1552

Sections 10.1: Sequences

Math 1552 lecture slides adapted from the course materials
By Klara Grodzinsky (GA Tech, *School of Mathematics*, Summer 2021)

Over the weekend, posted solutions to the following problems:

(1) $\displaystyle\int_0^1 \frac{\ln(x)}{\sqrt{x}}\,dx = -4$ (typo from $+4$)

(2) $\displaystyle\int_0^\infty \frac{e^{-\frac{1}{2x}}}{x^2}\,dx = 2$

(3) $\displaystyle\int_0^\infty \frac{e^x}{e^{2x}+3}\,dx = \frac{\pi\sqrt{3}}{9}$

(4) $\displaystyle\int_1^\infty \frac{dx}{x\sqrt{\ln(x)}} = +\infty$ (diverges)

Can you evaluate

$$I = \int_0^e \ln(x)\,dx \qquad \text{(show } I = 0\text{)?}$$

$\rightarrow$ make sure to review these notes before the midterm on Thursday.

Today's Learning Goals

- Use proper notation to denote a sequence.
- Understand how to find lower and upper bounds for sequences.
- Determine if a sequence is monotonic.
- Find limits of sequences when possible.

Sequences

A *sequence* is a *function* from the set of positive integers to the set of real numbers.

$$\{a_n\} = \{\cancel{f(n)}\} = \{a_1, a_2, \ldots, a_k, \ldots\}$$

$$\{a_n\} = \{a_1, a_2, a_3, \ldots\}$$

a_n is called the n^{th} term

OR

$$\{a_n\}_{n\geq 1} = \{a_1, a_2, a_3, \ldots\}$$

$$\{f(n)\}_{n=0}^{\infty} = \{f(0), f(1), f(2), \ldots\}$$

The values of *n* are all positive integers, unless otherwise specified, e.g., starting from n=0 in the third form above.

Example:

Find an expression for the general term of the sequence below:

$$\left\{-\frac{2}{3},\frac{3}{4},-\frac{4}{5},\frac{5}{6},\ldots\right\} = \{a_n\} = \{a_1, a_2, a_3, \ldots\}$$

A) $a_n = \dfrac{(-1)^n n}{n+1}$

B) $a_n = \dfrac{(-1)^{n+1} n}{n+1}$

C) $\boxed{a_n = \dfrac{(-1)^n (n+1)}{n+2}}$

D) $a_n = \dfrac{(-1)^{n+1}(n+1)}{n+2}$

$a_1 = -\frac{2}{3}$, $a_2 = \frac{3}{4}$, $a_3 = -\frac{4}{5}$,

$a_4 = \frac{5}{6}$

N	1	2	3	4	formula(N)
Sign of a_N	-1	$+1$	-1	$+1$	$(-1)^N$
Num. of a_N	2	3	4	5	$N+1$
denom of a_N	3	4	5	6	$N+2$

$\longrightarrow$ based on the first few values of the sequence:

$$a_N = (-1)^N \frac{(N+1)}{(N+2)}$$

LUB and GLB

$S = \{1, 2, 3, 4\}$, $M = 5$ is an upper bound for S

- An *upper bound* of a set S is a number *M* that is greater than or equal to each element in S.
- The smallest possible upper bound is called the *least upper bound* (*l.u.b.*) – *cf. the supremum*.

lub

$\text{lub}(S) = 4$

LUB and GLB

$S = \{1, 2, 3, 4\}$, $\underline{m} = 0$ is a lower bound for S

- An *upper bound* of a set S is a number *M* that is greater than or equal to each element in S.
- The smallest possible upper bound is called the *least upper bound* (*l.u.b.*) – *cf. the supremum*.
- A *lower bound* of a set S is a number *m* that is less than or equal to each element in S.
- The largest possible lower bound is called the *greatest lower bound* (*g.l.b.*) – *cf. the infimum.*

glb

$glb(S) = 1$

Example:

Find the l.u.b. and g.l.b. of the sequence: $\left\{\dfrac{n+1}{n}\right\}_{n\geq 1}$

$a_n = \dfrac{n+1}{n}$

$= \left\{\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots\right\} = S$

$\dfrac{n+1}{n} = 1 + \dfrac{1}{n}$, $\dfrac{1}{n+1} < \dfrac{1}{n}$ for $n \geq 1$

$a_{n+1} < a_n$ for $n \geq 1$

lub = least upperbound

$= a_1 = 2$

A. l.u.b.=1, g.l.b.=0

B. l.u.b.=2, g.l.b.=0

C. l.u.b.=2, g.l.b.=1

D. No l.u.b., g.l.b.=0

glb = greatest lower bound = ?

→ is 1 a lower bound for S? Yes.

→ if we pick glb = $1+\Delta$, $\Delta > 0$,

Then $1+\frac{1}{N} > 1+\Delta$, for

$$\frac{1}{\Delta} > N$$

→ So $\text{glb} = \lim_{N\to\infty} a_N = \lim_{N\to\infty} 1+\frac{1}{N} = 1$

Monotone Sequences

$\{1, 1, 2, 3, 3, 3, 4, \ldots\}$

A sequence is called ***monotonic*** if any one of the following statements holds:

$(i)\ a_n < a_{n+1}$ for all n (strictly increasing) *(cf. non-decreasing)*

$(ii)\ a_n \leq a_{n+1}$ for all n (monotonically increasing) $(*)$ *(cf. non-decreasing)*

$(iii)\ a_n > a_{n+1}$ for all n (strictly decreasing) *(cf. non-increasing)*

$(iv)\ a_n \geq a_{n+1}$ for all n (monotonically decreasing) *(cf. non-increasing)*

→ idea: (i) and (ii): subsequent terms of the sequence are at least as big as previous terms of the sequence.

Limit of a Sequence

$a_n = 1 + \frac{1}{n}$, for integer $n \geq 1$

graph:

$f(x) = 1 + \frac{1}{x}$

for all real $x \geq 1$

Let $\{a_n\}$ be a sequence. If $\lim_{n\to\infty} a_n = L$,

then L is the ***limit*** of this sequence.

If the sequence has a finite limit *L*, then the sequence is said to ***converge*** to L.

Otherwise, the sequence is said to ***diverge***.

$\{+1, -1, +1, -1, \ldots\} \longmapsto$ sequence diverges.

Convergence Theorem

$|a_n| \le B < \infty$ for all $n \ge 0$

If a sequence $\{a_n\}_{n\geq 0}$ is ***monotonic*** and ***bounded***,
then it converges (*to some finite limit L*).
If the sequence is *increasing*, then *L*=l.u.b. = least upper bound
If the sequence is *decreasing*, then *L*=g.l.b. = greatest lower bound

Equivalent statement:

An unbounded sequence diverges.

Example A: Determine whether the sequence converges. If so, find the limit. $\left\{\frac{n^2}{n+1}\right\}_{n\geq 1}$, $a_n = \frac{n^2}{n+1}$

$= \left\{\frac{1}{2}, \frac{4}{3}, \frac{9}{4}, \frac{16}{5}, \dots\right\} = S$

for all $n \geq 1$, $a_{n+1} > a_n$

$glb(S) = \frac{1}{2}$

$lub(S) = ?$, doesn't exist

→ The sequence increases monotonically without bound

→ in other words, it <u>diverges</u>

Example B:

Determine whether the sequence converges. If so, find the limit. $S = \{(-3)^n\}_{n\geq 1}$, $a_n = (-1)^n \cdot 3^n$

$S = \{-3, +9, -27, +81, \dots\}$

→ Sequence <u>diverges</u>!

→ every other term alternates in sign

→ $|a_{2n}| = 3^{2n} \longrightarrow +\infty$ as $n \to \infty$

$|a_{2n-1}| = 3^{2n-1} \longrightarrow +\infty$ as $n \to \infty$

Example C:

Determine whether the sequence converges. If so, find the limit. $S = \left\{ \frac{(-1)^n}{2^n} \right\}_{n \geq 1}$, $a_n = \frac{(-1)^n}{2^n}$

$S = \{ -\frac{1}{2}, +\frac{1}{4}, -\frac{1}{8}, +\frac{1}{16}, \ldots \}$

$\rightarrow$ sequence converges!

$\lim_{n \to \infty} a_n = 0$

$\rightarrow$ $glb(S) = -1/2$, $lub(S) = 1/4$

→ Note that a_n is <u>NOT</u> monotone!

Example D:

Determine whether the sequence converges. If so, find the limit. $S = \left\{ \dfrac{2^n}{n!} \right\}_{n \geq 1}$, $a_N = \dfrac{2^N}{N!}$

$S = \left\{ \dfrac{2}{1}, \dfrac{4}{2}, \dfrac{8}{6}, \dfrac{16}{24}, \dots \right\}$

$1! = 1$

$2! = 2 \cdot 1 = 2$

$3! = 3 \cdot 2 \cdot 1 = 6$

$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

$\rightarrow \lim_{N \to \infty} a_N = 0$

$\rightarrow a_{N+1} \leq a_N$ for all $N \geq 1$ (monotone)

$\rightarrow$ Sequence is bounded: $2 \geq a_N \geq 0$ for all N

$\rightarrow \operatorname{lub}(S) = a_1 = 2$

$\operatorname{glb}(S) = \lim_{n \to \infty} a_n = 0$

Example E: Determine whether the sequence converges. If so, find the limit. $S = \left\{ \sin\left(\frac{n\pi}{2}\right) \right\}_{n \geq 1}$

$\{1, 0, -1, 0, 1, 0, -1, 0, \ldots\} = S$

every 4th value repeats.

$\rightarrow \lim_{n \to \infty} a_n$ DNE (sequence diverges)

$\rightarrow glb(S) = -1$, $lub(S) = +1$

Example: Find the limit of the following sequence, if it exists: $S = \left\{ \dfrac{2n+1}{1-3n} \right\}_{n \geq 1}$ *(Justify your answer carefully.)*

$$a_n = \frac{-(2n+1)}{3n-1}, \quad n \geq 1$$

$$\to \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{-\frac{1}{n}(2n+1)}{\frac{1}{n}(3n-1)}$$

$$= \lim_{n\to\infty} \frac{-(2+\cancel{\frac{1}{n}}^{\to 0})}{(3-\cancel{\frac{1}{n}}^{\to 0})} = -2/3$$

A. 0

$\boxed{\text{B. } -2/3}$

C. 2/3

D. Diverges

Some Common Limits (memorize)

1) If $x > 0$, then $\lim_{n\to\infty} x^{1/n} = 1$.

2) If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$.

3) If $\alpha > 0$, then $\lim_{n\to\infty} \frac{1}{n^\alpha} = 0$.

$x=2$

4) $\lim_{n\to\infty} \frac{x^n}{n!} = 0$

5) $\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$

6) $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$

7) $\lim_{n\to\infty} n^{1/n} = 1$

8) If p is a positive integer, then:

$$\lim_{n\to\infty} \frac{a_p n^p + \cdots + a_1 n + a_0}{b_p n^p + \cdots + b_1 n + b_0} = \frac{a_p}{b_p}$$

(Do you see why?)

$\rightarrow$ can always multiply by $1 = \dfrac{\frac{1}{n^p}}{\frac{1}{n^p}}$

$\rightarrow$ before $p=1$, $\dfrac{-(2n+1)}{(3n-1)} \rightarrow -2/3$

$$\lim_{n\to\infty} (2n)^{1/n} = \lim_{n\to\infty} \underset{(1)}{2^{1/n}} \cdot \underset{(7)}{n^{1/n}} = 1 \cdot 1 = 1$$

Challenge problem on limits of sequences I:

Suppose that $a_n = \begin{cases} 1 + \frac{1}{a_{n-1}}, & \text{if } n \geq 1; \\ 1, & \text{if } n = 0. \end{cases}$

Does the sequence converge? If so, what is $\lim\limits_{n\to\infty} a_n$?

assume yes, then what is $L = \lim\limits_{N\to\infty} a_N$?

$$L = \lim_{N\to\infty}\left(1 + \frac{1}{a_{N-1}}\right) = 1 + \frac{1}{L}$$

$\Longleftrightarrow L^2 - L - 1 = 0$ (apply quadratic formula)

$$L = \frac{1 \pm \sqrt{5}}{2}$$

$\rightarrow$ but $\dfrac{1-\sqrt{5}}{2} < 0$ ✗

$\rightarrow$ so $L = \dfrac{1+\sqrt{5}}{2}$

Challenge problem on limits of sequences II:

(work this on your own)

Suppose that $b_n = \begin{cases} b_{n-1} + 2b_{n-2}, & \text{if } n \geq 2; \\ 1, & \text{if } n = 1; \\ 2, & \text{if } n = 0. \end{cases}$

Does the sequence converge? If so, what is $\cancel{\lim_{n\to\infty} b_n}$?

what is $\lim_{N\to\infty} \frac{b_N}{b_{N-1}}$?

<u>solution:</u> 2

Bonus problems on limits I:

L'Hopital's rule example

Evaluate the following limit: $$\lim_{n\to\infty}\left(1-\frac{x}{n}+\frac{x^2}{n^2}\right)^n = L$$

→ work this problem on your own

→ solution: $L = e^{-x}$

Bonus problems on limits II:

Evaluate the following limit: $\lim_{n\to\infty}\left(1+\frac{x^2}{n^2}\right)^{n^2} = L$

use L'Hopital's rule

→ work this problem on your own

→ <u>solution</u>: $L = e^{x^2}$

Math 1552

Sections 10.1: Review of Sequences

Math 1552 lecture slides adapted from the course materials
By Klara Grodzinsky (GA Tech, *School of Mathematics*, Summer 2021)

Review Question: Which of the following sequences converge?

$\checkmark$ (A) $\left\{\dfrac{2n+1}{1-3n}\right\} \xrightarrow{n\to\infty} -\dfrac{2}{3}$

$\times$ (B) $\left\{(-1)^n\right\} = \{-1, +1, -1, +1, \ldots\}$ (diverges)

$\checkmark$ (C) $\left\{\dfrac{2^n}{n!}\right\}$ (converges)

$\checkmark$ (D) $\left\{\left(1+\dfrac{4}{n}\right)^n\right\}$, $\lim\limits_{n\to\infty}\left(1+\dfrac{4}{n}\right)^n = e^4$ (converges)

Math 1552

Sections 10.2: Infinite Series

Math 1552 lecture slides adapted from the course materials
By Klara Grodzinsky (GA Tech, *School of Mathematics*, Summer 2021)

Learning Goals

- Understand what is meant by an infinite series
- Understand the general rule of when an infinite series converges
- Identify geometric series and find their sums
- Identify telescoping series and find their sums
- Determine convergence or divergence with the nth term test

Recall: Limit of a Sequence

Let $\{a_n\}$ be a sequence. If $\lim_{n\to\infty} a_n = L$,

then L is the ***limit*** of this sequence.

If the sequence has a finite limit *L*, then the sequence is said to ***converge*** to L.

Otherwise, the sequence ***diverges***.

includes the case when $\lim_{n\to\infty} a_n$ DNE

Review of Sigma Notation

Recall from the sections on Riemann sums that

$$\sum_{k=0}^{n} a_k = a_0 + a_1 + a_2 + \ldots + a_n$$

$$\sum_{k=1}^{n} 1 = n, \text{so} \sum_{k=0}^{n} 1 = n+1$$ (know this)

$m \leq n$ $$\sum_{k=m}^{n} (a_k + b_k) = \sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k$$ (linearity)

$m \leq n$ $$\sum_{k=m}^{n} c a_k = c \sum_{k=m}^{n} a_k$$ *(Linearity)*

$$\sum_{k=0}^{m} a_k + \sum_{k=m+1}^{n} a_k = \sum_{k=0}^{n} a_k$$ ~~*(Linearity)*~~ (change of index)

Infinite Series

An *infinite series* is a *sum* of infinitely many terms:

$$\sum_{k=0}^{\infty} a_k = a_o + a_1 + a_2 + a_3 + \ldots + a_n + \ldots.$$

$$= \lim_{N \to \infty} \sum_{k=0}^{N} a_k$$

Infinite Series

An *infinite series* is a *sum* of infinitely many terms:

$$\sum_{k=0}^{\infty} a_k = a_o + a_1 + a_2 + a_3 + \dots + a_n + \dots.$$

The series *converges* if the sequence of partial sums converges.

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=0}^{n} a_k = L \quad (L \text{ is finite})$$

The series *diverges* otherwise.

$$S_N = \sum_{k=0}^{N} a_k$$

Which of these series do you think converges?

(That is, à priori – we will cover precise criteria for each case in the next slides.)

(A) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ $= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\cdots$ (diverges)

(B) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+3)}$, $\frac{1}{(n+1)(n+3)} = \frac{1}{2}\left[\frac{1}{(n+1)} - \frac{1}{(n+3)}\right]$

$\rightarrow$ Converge (telescopes)

(C) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^{n}$ $= 1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots\cdots$ $= \frac{1}{1-2/3} = 3$

(geometric series: $r = \frac{2}{3} < 1$ converges)

(D) None of these